

PARTITIONING HADAMARD VECTORS INTO HADAMARD MATRICES

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ABSTRACT. We will show that in a space of dimension m , any family of 2^{m-1} distinct Hadamard vectors (where you can choose x or $-x$ but not both) can be partitioned into Hadamard matrices if and only if $m = 2^n$ for some n . We will solve this problem with a simple algorithm for assigning the vectors to the Hadamard matrices.

1. INTRODUCTION

Recall that

Definition 1.1. *In R^n , a vector of the form $x = (\pm 1, \pm 1, \dots, \pm 1)$ is called a **Hadamard vector**. An orthogonal matrix made up of Hadamard vectors is called a **Hadamard matrix**.*

Note that there are 2^n Hadamard vectors but they come as pairs $x, -x$ and so there are, up to sign, 2^{n-1} **distinct Hadamard vectors**. The **Hadamard Conjecture** states:

Hadamard Conjecture 1.2. *There exists a $(4n) \times (4n)$ Hadamard matrix for every n .*

These are the only possible cases since for $n = 2m + 1$, a maximal set of orthogonal Hadamard vectors contains just two vectors.

In this paper we will prove the following theorem.

Theorem 1.3. *In a space of dimension $m = 2^n$, any maximal set of distinct Hadamard vectors can be partitioned into $2^{2^n - n - 1}$ Hadamard matrices. Moreover, this result fails for all other values of m .*

2. SOME ELEMENTARY OBSERVATIONS

We make a few simple observations.

Observation 2.1. *If we can prove that just one choice of a maximal set of distinct Hadamard vectors can be partitioned into Hadamard matrices, then this is also true for any choice of distinct Hadamard vectors.*

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This is clear since any second choice of distinct Hadamard vectors has the property that for any vector x in this set, either x or $-x$ is in the first set. And so the partition of the first set is a partition of the second set with perhaps sign changes of some rows of the Hadamard matrix - which is still an orthogonal matrix.

Observation 2.2. *The moreover part of the theorem is essentially obvious.*

Given any $m = 4n$, if we can partition the 2^{4n-1} distinct Hadamard vectors into $(4n) \times (4n)$ Hadamard matrices, let k be the number of such matrices. Then $k \cdot (4n)$ uses up all the distinct Hadamard vectors and so

$$k \cdot (4n) = 2^{4n-1}, \text{ and so } m \text{ divides } 2^{4n-1}.$$

We will do the proof by induction on n with the case $n=2$ below:

$$A = \begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & - & + & - \\ + & - & - & + \end{bmatrix} \quad B = \begin{bmatrix} + & + & + & - \\ + & + & - & + \\ + & - & + & + \\ + & - & - & - \end{bmatrix}$$

3. PROOF OF THE THEOREM

We assume the theorem holds for $m = 2^n$ and we have partitioned a distinct set of Hadamard vectors into Hadamard matrices $\{A_i\}_{i=1}^{2^{2^n}-n-1}$ and let $\{x_{ij}\}_{j=1}^{2^n}$ be the row vectors of A_i . We will construct $2^{2^{n+1}-(n+1)-1}$ matrices of distinct Hadamard vectors of order $2^n \times 2^{n+1}$ so that cutting each of these matrices vertically in half, each of the left halves and the right halves are orthogonal matrices. For each of these, say $[A \ B]$, we then take:

$$(1) \quad \begin{bmatrix} A & B \\ A & -B \end{bmatrix}$$

and have a partition of distinct Hadamard vectors into Hadamard matrices. Since the total number of vectors here is equal to the total number of distinct Hadamard vectors for $2^{2^{n+1}}$, we are done. It will be obvious from our construction that the vectors we construct are unique.

For any n , we define the row shift of an $n \times n$ matrix A with row vectors $\{x_i\}_{i=1}^n$ by:

$$T_n = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \text{ so that } T_n \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_n \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

For the construction, for each A_i, A_j above, we form the $2^{2^n} \times 2^{2^n+1}$ matrices:

$$[A_i \mid T_m^k A_j] \text{ for all } k = 1, 2, \dots, 2^n.$$

Since the rows of A_i are orthogonal, the above matrices are pairs of orthogonal matrices and so are orthogonal. Note that each of the 2^{2^n-n-1} A'_i s is paired with all the other 2^{2^n-n-1} A'_j s and each is paired with 2^n shifts of the rows. So the total number of matrices above is:

$$2^{2^n-n-1} \cdot 2^{2^n-n-1} \cdot 2^n == 2^{2^{n+1}-(n+1)-1}.$$

I.e. We have used up all the distinct Hadamard vectors in a space of dimension 2^{n+1} .

REFERENCES

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